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We investigate when sampling a stochastic process $X = \{X(t) : t \geq 0\}$ at the times of an independent point process, ψ, leads to the same empirical distribution as the time average limiting distribution of X. Two cases are considered. The first is when X is an asymptotically stationary ergodic process and ψ satisfies a mixing and coupling condition. In this case, the entire limiting distributions in function space are shown to be the same. The second case is when X is only assumed to have a constant finite time average and ψ is assumed a positive recurrent renewal processes with a spread-out cycle length distribution. In this non-ergodic case, the averages are shown to be the same when some further conditions are placed on X and ψ.

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INDEPENDENT SAMPLING OF A STOCHASTIC PROCESS

by

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Independent Sampling of a Stochastic Process

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Abstract

We investigate when sampling a stochastic process $X = \{X(t) : t \geq 0\}$ at the times of an independent point process, ψ, leads to the same empirical distribution as the time average limiting distribution of X. Two cases are considered. The first is when X is an asymptotically stationary ergodic process and ψ satisfies a mixing and coupling condition. In this case, the entire limiting distributions in function space are shown to be the same. The second case is when X is only assumed to have a constant finite time average and ψ is assumed a positive recurrent renewal processes with a spread-out cycle length distribution. In this non-ergodic case, the averages are shown to be the same when some further conditions are placed on X and ψ.

Key Words: Time average, event average, independent sampling, asymptotically ergodic.

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1. Introduction

Quite a bit of literature has been devoted to establishing that under suitable conditions, the time average distribution of a stochastic process $X = \{X(t) : t \geq 0\}$ is the same as when averaging over the sampling times of an underlying point process $\psi = \{t_n : n \geq 0\}$. Formally this amounts to showing that for a set A in the state space of X,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t I(X(s) \in A)ds = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} I(X(t_n) \in A), \tag{1.1}$$

where $I(X(s) \in A)$ denotes the indicator function for the event $\{X(s) \in A\}$.

The main emphasis in the literature has dealt with the case when X and ψ are dependent; for example when t_n is the arrival time of the n^{th} customer to a queueing system (with which each arrival interacts), and $X(t)$ is the state of the system at time t. The classic and fundamental result in this regard is PASTA (Poisson Arrivals See Time Averages) (Wolff[13]) which states that under a so-called *Lack of Anticipation Property*, sampling by a Poisson process does the trick. In such cases, path regularity assumptions (such as left or right continuity) are placed on X because $X(t_n-)$ need not be equal to $X(t_n+)$.

Many papers generalizing, extending and giving converses to PASTA have appeared in recent years, giving rise to the general notion of ASTA (Arrivals See Time Averages) (Miyazawa and Wolff [10], Melamed and Whitt[8],[9], Green and Melamed[5], Wolff[14], Konig, D. and V. Schmidt [6], Bremaud [2], Bremaud et. al.[3]). Nevertheless, it seems that perhaps the *easiest case* has not been seriously studied: the case when X and ψ are independent (but not necessarily stationary).

The purpose of the present paper is to fill in this gap. We don't assume any path regularity assumptions for X nor do we assume that X or ψ are stationary processes. The problem turns out to be more difficult than one might expect. We consider two set-ups. In the first (section 2) X is assumed asymptotically stationary ergodic (ASE) and ψ is assumed a point process that admits coupling to a stationary version and is mixing. We show that the sampled process has the same limiting distribution (Theorem 2.1.). The distributions we deal with are those in function space (not just the *marginal* distribution as in (1.1)). As a corollary (Corollary (2.1)), it is seen that if X is ASE and ψ is a positive recurrent renewal process with a spread-out cycle length distribution then the result holds. Finally, we give a general converse that does not require any further conditions (Theorem 2.2), and counterexamples (Remark(2.2)) showing that neither a non-lattice renewal process nor a stationary one nor one with a smooth delay cycle is sufficient to obtain Corollary (2.1).

In the second set-up (section 3) we are no longer interested in equating *distributions* as in (1.1) but only the average of a real-valued process:

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X(s)ds = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X(t_n). \tag{1.1}$$

For example, X could be of the form $X(t) = I\{Y(t) \in A\}$ where Y has a general state space and A is a fixed set of states. Or X could be a deterministic real valued function $X(t) = x(t)$ with a finite Cesaro limit. Therefore, in this case, X is not assumed asymptotically stationary ergodic. We assume, though, that the point process is renewal with a spead-out cycle length distribution. A further moment condition is also assumed.

2. The Asymptotically Ergodic Case and a General Converse

Let $X = \{X(t) : t \geq 0\}$ be a stochastic process on some underlying probability space $(\Omega, \mathcal{B}, P)$, with $X(t)$ taking values in the *state space* $\mathcal{S}$ (a measure space endowed with σ-field $\mathcal{F}$). We assume that $X(t,w)$ ($t \geq 0$, $w \in \Omega$) is jointly measurable and view X as a random element of the function space $\mathcal{K} \stackrel{\text{def}}{=} \mathcal{S}^{\mathcal{R}_+}$ endowed with the product σ-field.

We wish to *sample* X at the times of an independent point process $\psi = \{t_n : n \geq 0\}$. To start with, we assume that X and τ are on the same probability space, and that ψ is *simple*, that is, that the t_n are strictly increasing (to infinity) as $n \longrightarrow \infty$.

We view ψ as a random point measure on $\mathcal{R}_+$, where for any Borel set $A \subset \mathcal{R}_+$,

$$\psi(A) = \sum_{n=0}^{\infty} I\{t_n \in A\}. \tag{2.1}$$

We let $\psi(t) \stackrel{\text{def}}{=} \psi((0,t])$ denote the associated counting process.

$\mathcal{M}$ denotes the space of all point measures, $\mu = \mu(\cdot)$, that are bounded on compact intervals, equiped with the $\hat{w}$ topology (and associated Borel sets) defined via: $\mu_k \longrightarrow \mu$ as $n \longrightarrow \infty$ iff for each bounded continuous function, $f : \mathcal{R}_+ \longrightarrow \mathcal{R}$, with compact support, $\mu_k(f) \longrightarrow \mu(f)$. This makes $\mathcal{M}$ into a complete separable metric space (consult for example page 628 of Daley and Vere-Jones[4]). ψ is assumed a random element of $\mathcal{M}$.

For $s \geq 0$, $\theta_s : \mathcal{K} \rightarrow \mathcal{K}$ denotes the *shift operator* $(\theta_s x)(t) = x(s+t)$, and it also will be used to denote the shift on $\mathcal{M}$; $\theta_s \psi(A) = \psi(s + A)$.

We say that X is *asymptotically stationary ergodic* (ASE) with respect to the time shifts $\{\theta_s\}$, if there exists a *limiting probability distribution*,Q, on $\mathcal{K}$ in the sense that for all measurable sets $A \subset \mathcal{K}$

$$Q(A) = \lim_{t \to \infty} \frac{1}{t} \int_0^t I(\theta_s \circ X \in A) ds, \quad a.s.P. \tag{2.2a}$$

The measure Q is necessarily stationary and ergodic with respect to the shifts and if X^* denotes a process with distribution Q, $Q(A) = P(X^* \in A)$, then X^* is a time stationary ergodic process.

ASE is equivalent to (2.2a) holding for all non-negative measurable functions $f : \mathcal{K} \longrightarrow \mathcal{R}_+$ (see page 101 of Loeve [7]):

$$Q(f) = \lim_{t \to \infty} \frac{1}{t} \int_0^t f(\theta_s \circ X) ds; \quad a.s.P, \tag{2.2b}$$

and is also equivalent to: for all measurable sets A_0, A

$$P(X \in A_0)Q(A) = \lim_{t \to \infty} \frac{1}{t} \int_0^t P(X \in A_0, \theta_s X \in A) ds. \tag{2.2c}$$

ASE holds if (for example) X is *positive recurrent regenerative* or *stationary ergodic* or is a positive *Harris recurrent Markov process.*

We say that ψ is *mixing* if for all Borel sets B_0 and B,

$$P(\psi \in B_0, \theta_s\psi \in B) \longrightarrow P(\psi \in B_0)M(B), \quad \text{as} \quad s \longrightarrow \infty, \tag{2.3}$$

where M denotes a probability measure on $\mathcal{M}$. Mixing implies ergodicity (via (2.2c)); so in particular M is a stationary ergodic measure with respect to the shifts and letting ψ^* denote a random point process with distribution M, ψ is stationary ergodic. It is easily seen that ψ^* is also mixing. $\lambda \stackrel{\text{def}}{=} E\psi^*(1)$ denotes the *intensity* of ψ which we assume is finite and non-zero.

We say that ψ *admits coupling* to ψ^* if there exists versions of ψ and ψ^* on the same probability space together with a proper random time $\mathcal{T}$ such that $\theta_s\psi = \theta_s\psi^*; \quad s \geq \mathcal{T}$.

Theorem 2.1. *Suppose X is ASE with limiting distribution Q from (2.2a). If τ is independent of X and is mixing and admits coupling to ψ^* then sampling by ψ leads to the same limiting distribution:*

$$Q(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} I(\theta_{t_k} \circ X \in A) \quad a.s.P. \tag{2.4}$$

Proof: Consider the joint random element $Z = (X, \psi)$ on the product space $\mathcal{K} \times \mathcal{M}$. It suffices to prove the theorem for stationary version $\psi^* = \{t_n^*\}$, for suppose (2.4) holds for (X, ψ^*). Then by using the coupling time T, we can define discrete random times $S_1 \stackrel{\text{def}}{=} \min\{n : t_n \geq T\}$, and $S_2 \stackrel{\text{def}}{=} \min\{n : t_n^* \geq T\}$, and deduce that $t_{S_1+n} = t^*_{S_2+n}, n \geq 0$, implying that (2.4) will hold for Z. For the rest of the proof, we assume that $\psi = \psi^*$.

We shall first show that Z is jointly ASE with limiting distribution $Q \times M$. This is equivalent to showing that the stationary measure $Q \times M$ is ergodic and that for all measurable sets C in the product σ-field (of $\mathcal{K} \times \mathcal{M}$)

$$(Q \times M)(C) = \lim_{t\to\infty} \frac{1}{t} \int_0^t P(\theta_s(X, \psi) \in C)ds. \tag{2.5}$$

To establish (2.5), define for each $y \in \mathcal{M}$ the *section* $A(y) \stackrel{\text{def}}{=} \{(x, y) \in \mathcal{K} \times \mathcal{M} : (x, y) \in C\}$. Then by Fubini's Theorem and independence, for each t

$$\frac{1}{t} \int_0^t P(\theta_s(X, \psi) \in C)ds = \int_{\mathcal{M}} \frac{1}{t} \int_0^t P(\theta_s X \in A(y))M(dy). \tag{2.6}$$

By the Dominated Convergence Theorem, (2.6) converges as $t \longrightarrow \infty$ to

$$\int_{\mathcal{M}} Q(A(y))M(dy) = (Q \times M)(C), \tag{2.7}$$

the last step once again due to Fubini's Theorem.

$Q \times M$ is indeed stationary, but to prove ergodicity (via (2.2c)), letting (X^*, ψ) have this joint stationary distribution , it suffices to show (see Lemma 10.3.II, page 341 of Daley and Vere-Jones[4]) that for all measurable *cylinder* sets $(A_0, B_0), (A, B)$

$$Q(A_0)Q(A)M(B_0)M(B) = \lim_{t \to \infty} \frac{1}{t} \int_0^t P(X^* \in A_0, \theta_s X^* \in A) P(\psi \in B_0, \theta_s \psi \in B) ds. \tag{2.8}$$

This follows easily from (2.3) and the ergodicity of Q.

Thus (via (2.2b) we have that for all measurable $f : \mathcal{K} \times \mathcal{M} \longrightarrow \mathcal{R}_+$

$$(Q \times M)(f) = \lim_{t \to \infty} \frac{1}{t} \int_0 f(\theta_s \circ X) ds; \quad a.s.P. \tag{2.9}$$

In (2.9), choose, for fixed Borel set A, the function

$$f(X, \psi) = \lambda^{-1} \sum_{n=0}^{\psi(1)} I\{\theta_{t_n} X \in A\}, \tag{2.10}$$

so that the limit in (2.9) is

$$(Q \times M)(f) = E(f(X^*, \psi)) = \lambda^{-1} E \sum_{n=0}^{\psi(1)} I\{\theta_{t_n} X^* \in A\} = Q(A), \tag{2.11}$$

the last equality due to the independence and stationarity of X^* and ψ. Moreover,

$$\int_0^t f(\theta_s X, \theta_s \psi) ds = \lambda^{-1} \int_0^t \sum_{n=\psi(s)}^{\psi(s+1)} I\{\theta_{t_n} X \in A\} ds = \lambda^{-1} \sum_{n=0}^{\psi(t+1)} I\{\theta_{t_n} X \in A\} \int_0^t I\{t_j \in (s, s+1]\} ds. \tag{2.12}$$

The last integral tends to 1 as $t \longrightarrow \infty$ and $(\psi(t+1) - \psi(t))/t \longrightarrow 0$, so we obtain

$$Q(A) = \lim_{t \to \infty} \lambda^{-1} \frac{1}{t} \sum_{n=0}^{\psi(t)} I\{\theta_{t_n} X \in A\}. \tag{2.13}$$

Since $\psi(t)/t \longrightarrow \lambda$, we finally obtain (2.4). ■

A distribution F on $\mathcal{R}_+$ is said to be *spread-out* if for some integer $m \geq 1$, the convolution $F \star \cdots \star F$, m times, has an absolutely continuous component with respect to Lebesgue measure.

Corollary 2.1. *Suppose τ is a positive recurrent renewal process, independent of X and has a spread-out cycle length distribution. Then if X is ASE with limiting distribution Q, (2.4) holds.*

Proof: For each fixed t, let $\mathcal{F}_t$ denote the σ-field generated by the point measures restricted to $[0, t]$: $\{\mu(s) : s \leq t\}$. The spread-out condition is both the sufficient and necessary one for renewal process ψ to admit coupling to a stationary version ψ^* and do so for any initial condition of the form $\psi \in \mathcal{F}_t$. (an immediate consequence of the corresponding result concerning the Markov process of forward recurrence times for ψ (see for example Theorem 2.3, page 146 in Asmussen[1])). Similarly, from this, it also follows that $\theta_s \psi$ converges in total variation to limiting distribution M egardless of such initial conditions (see Corollary 1.5 page 142 of [1]). In particular, for each fixed measurable B, mixing condition (2.3) holds for any $B_0 \in \bigcup_{t \geq 0} \mathcal{F}_t$. We can assume without loss of generality that ψ is stationary (via coupling) and thus since $\bigcup_{t \geq 0} \mathcal{F}_t$ forms a semiring that generates the Borel sets of $\mathcal{M}$, the proof is complete by Lemma 10.3.II in [4]. ■

We now consider the converse of Theorem 2.1 (i.e. we assume (2.4) and try to deduce (2.2a)) and find that we do not need the mixing condition on ψ nor any ergodicity condition on X.

Theorem 2.2. *If X is a stochastic process and τ is a point process independent of X that admits coupling to a stationary version ψ^*, such that for some fixed measurable set $A \subset \mathcal{K}$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} I(\theta_{t_k} \circ X \in A) = \alpha \quad a.s.P, \tag{2.14}$$

with α a constant, then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t I(\theta_s \circ X \in A)ds = \alpha, \quad a.s.P. \tag{2.15}$$

Proof: If (2.14) holds for (ψ, X), then it will also hold for (ψ^*, X) by coupling ψ to ψ^*. Thus it suffices to prove (2.15) in the stationary case. From (2.14) it follows that as $t \longrightarrow \infty$

$$\frac{1}{t}\sum_{k=1}^{\psi(t)} I(\theta_{t_k} \circ X \in A) \longrightarrow \lambda\alpha \quad a.s.P.$$

Since the summation is bounded above by the $\frac{\psi(t)}{t}$, which is Uniformly Integrable (UI) (due to stationarity),

$$\frac{1}{t}\int_0^t I(\theta_s \circ X \in A)\lambda ds = E\{\frac{1}{t}\sum_{k=1}^{\psi(t)} I(\theta_{t_k} \circ X \in A)|X\} \longrightarrow \lambda\alpha \quad a.s.P$$

Dividing by λ gives (2.15). ■

Remark(2.1): Another way of understanding Theorem (2.1) is through *Palm* theory : Under the hypothesis of the theorem, the time stationary distribution of (X, ψ) is $Q \times M$ and the sampled stationary distribution is the *Palm distribution* of $Q \times M$ (see for example Rolski[11]) which in this case is precisely $Q \times M^0$ where M^0 is the Palm distribution of M. The Palm distribution is defined via

$$(Q \times M)^0(f) = \lambda^{-1} E_{Q\times M} \sum_{n=0}^{\psi(1)} f(\theta_{t_n} X, \theta_{t_n} \psi),$$

and denotes the limiting distribution obtained by averaging the distribution of (X, ψ) over all the sampling times t_n. The Palm distribution is stationary ergodic with respect to the shifts θ_{t_k}; in particular, under M^0, the interarrival times $\{t_n - t_{n-1}\}$ form a stationary ergodic sequence.

Remark(2.2): A non-lattice cycle-length distribution, F, is not enough to ensure Corollary 2.1 : Define a cyclic deterministic process $X(t) = t - n; \quad n \le t < n + 1$, (the fractional part of t). Let A denote the set of irrational numbers in $(0, 1)$ and take F to have mass only on the rationals (with positive mass on each rational). Then the time average of $I\{X(s) \in A\}$ is 1 but the sampled average is 0. In fact, when F is not spread-out, then by only smoothing out the first (i.e. the delay) cycle length (making it spread-out) or choosing a stationary renewal process, Corollary 2.1 once again is false: Break up $\mathcal{R}_+$ into the odd half intervals $[n, n+1/2)$, and even half intervals $[n+1/2, n+1]$, $n \geq 0$, and define $X(t)$ to be 1 on the even ones and 0 on the odd ones so that the time average of $\{X(s) = 1\}$ is .5 a.s. For sampling, take a deterministic renewal process with interevent times identically 1, but let t_1 have a $Unif(0,1)$ distribution (this results in making the renewal process time stationary). Then with probability .5, the event $\{t_1 \leq .5\}$ will occur in which case $X(t_n) = 0$ for all n, giving the event average as 0 w.p. .5.

3. The Non-Ergodic Case With Renewal Sampling

In section 2 we assumed that X was asymptotically ergodic which then allowed us to use ergodic theory to deduce our desired result. Suppose, however, that for a stochastic process X we only know that for some fixed *marginal* Borel set $A \subset \mathcal{S}$ that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t I\{X(s) \in A\}ds = \alpha, \quad a.s.P. \tag{3.1}$$

where α is a finite constant. X need not be AE to satisfy (3.1) so, in this section, we investigate sufficient conditions ensuring that

$$\alpha = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} I\{X(t_k) \in A\} \quad a.s.P, \tag{3.2}$$

when ψ is independent of X.

We shall actually be interested in obtaining the more general result of equating averages for a real-valued process X. In this case

$$\alpha = \lim_{t\to\infty} \frac{1}{t}\int_0^t X(s)ds, \quad a.s.P, \tag{3.3}$$

is assumed to exist and we wish to give sufficient conditions ensuring that

$$\alpha = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X(t_k) \quad a.s.P. \tag{3.4}$$

Whereas one would expect (3.4) to hold under fairly general conditions, we no longer have ergodic theory at our disposal and hence must resort to a different approach which for us requires from the start assuming that ψ is renewal with a spread-out cycle length distribution. Let T denote a generic cycle length.

Theorem 3.1. *Suppose $X = \{X(t)\}$ is a jointly measurable bounded process for which (3.3) holds for finite constant α. If ψ is a renewal process independent of X, with a spread-out cycle length distribution, satisfying $ET^{1+\epsilon} < \infty$, for some $\epsilon > 0$ then (3.4) holds.*

Proof: Assume $\sup_t X(t) \leq M < \infty$, and Let ψ^* denote a time stationary version of ψ (to which, due to the spread-out assumption ψ admits coupling). It suffices to show that

$$\lambda\alpha = \lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{\psi^*(t)} X(t_k^*) \quad a.s.P. \tag{3.5}$$

Let $U(t) = E\psi(t)$ denote the renewal measure for ψ. First observe that

$$E \sum_{k=1}^{\psi^*(t)} X(t_k^*) = \int_0^t EX(s)E\psi^*(ds) = \lambda \int_0^t EX(s)ds, \tag{3.6}$$

and that similarly (see page 136 of Daley and Vere-Jones[4])

$$E\{\sum_{k=1}^{\psi^*(t)} X(t_k^*)\}^2 = \lambda \int_0^t EX^2(s)ds + 2\lambda E \int_0^t \int_0^{t-s} X(s+u)U(du)X(s)ds. \tag{3.7}$$

Consequently

$$\begin{aligned} Var\{\sum_{k=1}^{\psi^*(t)} X(t_k^*)/t\} &= \frac{\lambda}{t^2} \int_0^t EX^2(s)ds + 2\frac{\lambda}{t^2} E \int_0^t \int_0^{t-s} X(s+u)(U(du) - \lambda du)X(s)ds \\ &\leq \lambda \frac{M^2}{t} + 2\frac{\lambda^2 M^2}{t^2} \int_0^t \sum_{k=0}^{\lfloor t-s \rfloor} (|U(du) - \lambda du|(k + [0,1])). \end{aligned} \tag{3.8}$$

Using the hypothesis that $ET^{1+\epsilon} < \infty$, we apply Theorem 1 of Stone and Wainger[12] to deduce that the last integral in (3.8) reduces to

$$\int_0^t \sum_{k=0}^{\lfloor t-s \rfloor} o(k^{-\epsilon})ds, \tag{3.9}$$

yielding (after integration)

$$Var\{\sum_{k=1}^{\psi^*(t)} X(t_k^*)/t\} \leq \lambda \frac{M^2}{t} + 2\frac{\lambda^2 M^2}{t^2} O(t^{2-\epsilon}), \tag{3.10}$$

and finally

$$Var\{\sum_{k=1}^{\psi^*(t)} X(t_k^*)/t\} = O(t^{-\epsilon}). \tag{3.11}$$

Using the susequence $s_n \stackrel{\text{def}}{=} n^{\frac{2}{\epsilon}}$ in (3.11) we therefore obtain

$$Var\{\sum_{k=1}^{\psi^*(s_n)} X(t_k^*)/s_n\} = O(\frac{1}{n^2}). \tag{3.12}$$

Applying Chebychev's inequality while using Borel-Cantelli, yields

$$\frac{1}{s_n} \sum_{k=1}^{\psi^*(s_n)} X(t_k^*) - \lambda \frac{1}{s_n} \int_0^{s_n} EX(s)ds \longrightarrow 0, \quad a.s. \tag{3.13}$$

as $n \longrightarrow \infty$. By bounded convergence, $\frac{1}{s_n}\int_0^{s_n} EX(s)ds \longrightarrow \alpha$, hence by (3.13)

$$\frac{1}{s_n}\sum_{k=1}^{\psi^*(s_n)} X(t_k^*) \longrightarrow \alpha \quad a.s. \tag{3.14}$$

Let $n(t)$ denote the (deterministic) counting process for $\{s_n\}$. To prove (3.5) from (3.13), observe that for any t, there exists an $n(t)$ such that

$$n(t)^{2/\epsilon} \leq t \leq (n(t)+1)^{2/\epsilon}, \tag{3.15}$$

so

$$\begin{aligned} \left|\frac{1}{t}\sum_{k=1}^{\psi^*(t)} X(t_k^*) - \frac{1}{t_{n(t)}}\sum_{k=1}^{\psi^*(t_{n(t)})} X(t_k^*)\right| &\leq \left|\frac{1}{t} - \frac{1}{t_{n(t)}}\right|\sum_{k=1}^{\psi^*(t)} |X(t_k^*)| + \frac{1}{t_{n(t)}}\sum_{k=\psi^*(t_{n(t)+1})}^{\psi^*(t)} |X(t_k^*)| \\ &\leq M\frac{\psi^*(t)}{t}\frac{[(n(t)+1)^{2/\epsilon} - n(t)^{2/\epsilon}]}{n(t)^{2/\epsilon}} + \frac{1}{n(t)}\sum_{\psi^*(t_{n(t)+1})}^{\psi^*(t)} |X(t_k^*)|, \end{aligned} \tag{3.16}$$

the second to last piece of which tends to zero. But the same argument as used above can be used on $|X(t)|$, implying that the last piece in (3.16) also tends to zero and thus proving (3.5). ■

Remark(3.1): If X is not bounded, then Theorem 3.1 may fail even for Poisson arrivals as the following counterexample shows: Define $X(t) = n^2$ for $t \in [n, n+1/n^2)$, $X(t) = 0$ otherwise. Then the limit in (3.3) is 1 whereas the limit in (3.4) is 0.

Remark(3.2): When X is a bounded process, then Theorem 2.2 extends to averages as in (3.4).

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